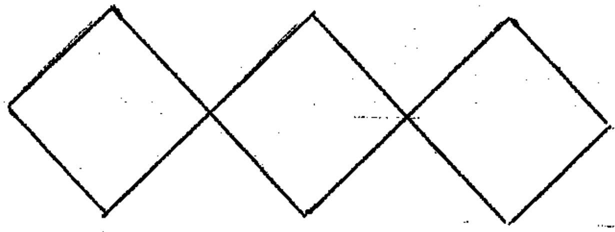
S. I. Jones
Mathematical Nuts

EVENT 5A SOLUTIONS

1.



2.

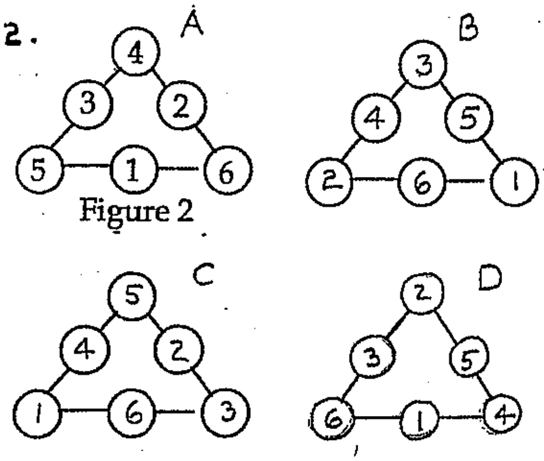


Figure 2

Note that if each number in A is subtracted from 7, one obtains B. Similarly compare C and D.

Any two of patterns B, C, or D are acceptable.

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3. The total area of the three circles is $\pi 3^2 + \pi 4^2 + \pi 5^2 = 50\pi$, so each share should be $\frac{50\pi}{4} = \frac{25\pi}{2}$, which is half of the larger circle. Cut from the middle circle a circle equal to the small circle, and then divide the outer ring equally between A and B.

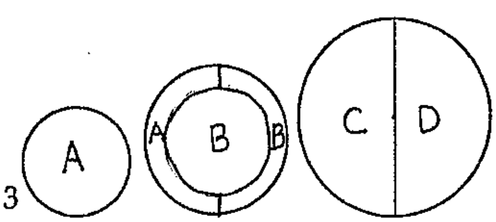


Figure 3

Proceed by back substitution.

$$\begin{aligned}
a &= e+11 && = e+11\\
b &= a-19 = (e+11)-19 && = e-8\\
c &= b+7 = (e-8)+7 && = e-1\\
d &= c+5 = (e-1)+5 && = e+4\\
+\ e & && = e\\
\hline
56 & && = 5e+6
\end{aligned}$$

$\boxed{e = 10}$

4. Let a, b, c, d, e represent the dollars of the five shoppers. We are told that

$$a+b+c+d+e = 56$$

and that one of each of the following pairs of equations is true.

$$\begin{aligned}
a-b &= \pm 19\\
b-c &= \pm 7\\
c-d &= \pm 5\\
d-e &= \pm 4\\
e-a &= -11
\end{aligned}$$

AHSME 1997 #22

Since the sum of the left sides of these equations is 0, we must choose one number from each of $\{19,-19\}, \{7,-7\}, \ldots, \{11,-11\}$ so that their sum is 0.

The possibilities are:

19, −11, −7, −5, 4

or

−19, 11, 7, 5, −4

Consider the system

$$\begin{aligned}
a-b &= 19\\
b-c &= -7\\
c-d &= -5\\
d-e &= 4\\
e-a &= -11
\end{aligned}$$

Proceed in the same way with the alternate system

$$\left.\begin{aligned}
a-b &= -19\\
b-c &= 7\\
c-d &= 5\\
d-e &= -4\\
e-a &= 11
\end{aligned}\right\}\text{ This system does not give an integer value for e.}$$

Solutions

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 5B

5 4 The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

13.8

$8\sqrt{3}$ 1. The rhombus *ABCD* pictured in Figure 1 has sides of length 4, and the altitude dropped from *D* bisects *AB*. What is the area of *ABCD*?

$4\sqrt{3}$ 6.928 2. The mid-points of the rhombus described in Problem 1 are joined to form an inscribed quadrilateral. What is its area?

$\frac{x^2}{x^2-2x+2}$ 3. In the square *ABCD* with sides of length 1 shown in Figure 3, let $x = AE = BF = CG = DH$. Find the area of *RSTU* in terms of x.

$\frac{1}{2}\,\frac{1-x}{x^2-2x+2}$ 4. Using the square and the notation introduced in Problem 3, find the area of ΔBCS in terms of x.

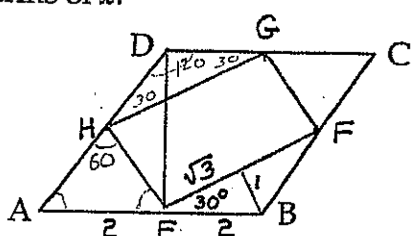


Figure 1

1. Δ AED is a 30-60-90 triangle.
$\therefore$ $DE = AE\sqrt{3} = 2\sqrt{3}$
Area (ABCD) $= 4\,(2\sqrt{3}) = 8\sqrt{3}$

2. Δ AEH is equilateral; $HE = 2$.
Base angles of isosceles Δ HGD are $30°$
$\angle GHE = 180 - (60+30) = 90$
$\therefore$ EFGH is a rectangle with $EF = 2\sqrt{3}$
Area (EFGH) $= (HE)(EF) = 2\,(2\sqrt{3})$

3.

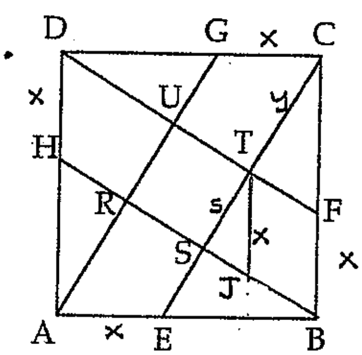


Figure 3

slope AG $= \frac{1}{1-x}$

slope BH $= \frac{1-x}{-1}$

$\therefore$ AG $\perp$ BH

Similarity arguments show then that RSTU is square.

Let s = length of a side of RSTU
From T, drop TJ $\parallel$ CB
$\Delta STJ \sim \Delta BCE \Rightarrow \frac{TS}{CB} = \frac{TJ}{CE}$
i.e. $\frac{s}{1} = \frac{x}{\sqrt{(1-x)^2+1}}$
Area (RSTU) $= s^2 = \frac{x^2}{x^2-2x+2}$

4. Let $y = CT$. $\Delta SCB \sim \Delta STJ$ so $\frac{y+s}{1} = \frac{s}{x}$
Solve for y, using $s = \frac{x}{\sqrt{x^2-2x+2}}$
$y = \frac{1}{x}\,s\,(1-x) = \frac{1-x}{\sqrt{x^2-2x+2}}$. Area $(\Delta BCS) = \frac{1}{2}(s+y)y = \frac{1}{2}\,\frac{1-x}{x^2-2x+2}$

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 5C

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

5005 1. In the expansion of $(x+y)^{15}$, what is the coefficient of x^9y^6?

220 2. The Screening Committee is to present to the President a list, in alphabetical order, of the names of three acceptable candidates. How many different lists are possible, given that the names of the candidates are: Abdullah, Bjorn, Costillo, Diem, Ellsworth, Fiorello, Gin, Hu, Ione, Juarz, Klein, Longley.

$\frac{5}{13}$ .385 3. A bag contains 4 red and 8 white marbles, well mixed. One marble is removed and replaced by two marbles of the other color. Again after thorough mixing, a marble is drawn. What is the probability that this last marble to be removed is red?

$\frac{59}{125}$ .472 4. if a, b, and c are three (not necessarily different) integers chosen randomly and with replacement from the set {1. 2. 3. 4. 5}, what is the probability that $ab+c$ is even?

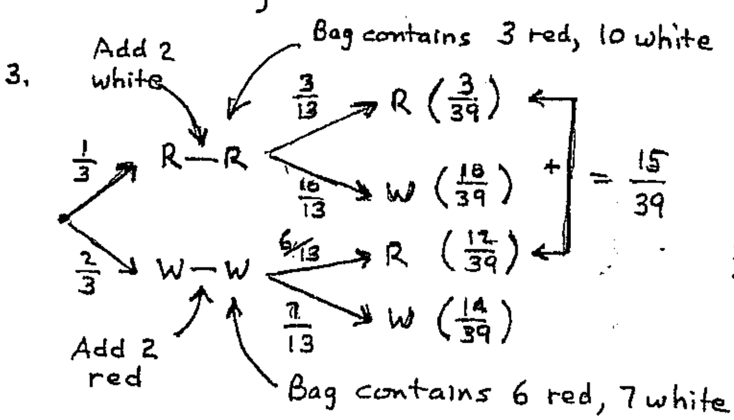
1. The coefficient is

$$\binom{15}{9} = \frac{15\cdot 14\cdot 13\cdot 12\cdot 11\cdot 10}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1} = 5005$$

2. There are $\binom{12}{3} = \frac{12\cdot 11\cdot 10}{3\cdot 2} = 220$

ways to choose 3. Alphebetize after they are chosen

3.



AHSME 1995 #20

4. $ab+c$ will be even if

(I) ab and c are both even

or

II ab and c are both odd.

ab is odd only if a and b are odd; prob $= \frac{3}{5}\cdot\frac{3}{5} = \frac{9}{25}$

ab is even if not odd, so prob $= 1 - \frac{9}{25} = \frac{16}{25}$

Case I $\quad \frac{16}{25}\cdot\frac{2}{5} = \frac{32}{125}$

Case II $\quad \frac{9}{25}\cdot\frac{3}{5} = \frac{27}{125}$

prob (even) $= \frac{32+27}{125} = \frac{59}{125}$

Solutions

Minnesota State High School Mathematics League

Individual Event

2006-07 Event 5D

Questions in this event are written, with permission, as variations of problems from the 2006 AMC-12 Exam. Review of this exam is excellent preparation, not only for this event, but for the 2007 AMC-12, which we strongly encourage you to take.

The first question is intended to be a quickie and is worth 1 point. Each of the next three questions is worth 2 points. Place your answer to each question on the line provided. You have 12 minutes for this event.

__1__ 1. For any two real numbers x and y, define $x \oplus y = \frac{x+y}{2}$. Find the numeric value of $[(3 \oplus 5) \oplus 7] - [3 \oplus (5 \oplus 7)]$.

355,000

__113π__ 2. The vertices of a 5-12-13 right triangle are centers of three mutually externally tangent circles as shown in Figure 2. What is the sum of the areas of these circles?

Accept any notation that indicates a domain of one point

__$x=1$__ 3. The function f has the property that for each real number x in its domain, $\frac{1}{x}$ is also in its domain, and $f(x) + f\left(\frac{1}{x}\right) = \frac{1-x}{1+x}$. What is the largest set of numbers that can be in the domain of f?

$(1004)^2$ or 1,008,016

______ 4. The expression $(x+y+z)^{2007} + (x-y-z)^{2007}$ is expanded and simplified by combining like terms. How many terms are in this simplified expression?

1. $\frac{3+5}{2} \oplus 7 = \frac{4+7}{2} = \frac{11}{2}$

$3 \oplus \frac{5+7}{2} = \frac{3+6}{2} = \frac{9}{2}$

$\frac{11}{2} - \frac{9}{2} = 1$

2. Using s, m, and ℓ as radii of the small, medium, and large circles, solve

$s + m = 5$

$s + \ell = 12$

$m + \ell = 13$

$s = 2;\ m = 3;\ \ell = 10$

Area $= \pi(4 + 9 + 100)$

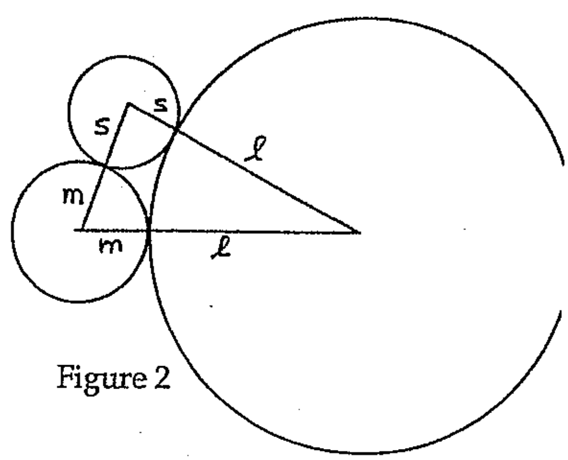


Figure 2

3. $f\left(\frac{1}{x}\right) + f\left(\frac{1}{1/x}\right) = \dfrac{1-\frac{1}{x}}{1+\frac{1}{x}} = \frac{x-1}{x+1}$

$f\left(\frac{1}{x}\right) + f(x) = -\frac{1-x}{x+1}$

$= -\left[f(x) + f\left(\frac{1}{x}\right)\right]$

$2\left[f(x) + f\left(\frac{1}{x}\right)\right] = 0 = 2\frac{x-1}{x+1}$

$\therefore\ x = 1$

4. For each term $kx^a y^b z^c$, there is a term $kx^a(-y)^b(-z)^c$. These drop out if b and c are opposite in parity. Since $a+b+c = 2007$, terms remain only if a is odd.
Total choices $= (2008-1) + (2008-3) + \cdots + 1 = (1004)^2$

Minnesota State High School Mathematics League

Team Event

2006-07 Meet 5

Each question is worth 4 points. Team members may cooperate in any way, but at the end of twenty minutes, one set of answers is to be submitted. Put answers on the lines provided.

1. In Figure 1, place the integers 1, 2, 3, 4, 5, 6, 7, 8 in the circles along the edges of the square so that the sum of the integers on each side is the same. Do the same in each of the remaining squares, taking care to get sums that are the same on each side of a square, but different for each square. You get 1 point for each square completed.

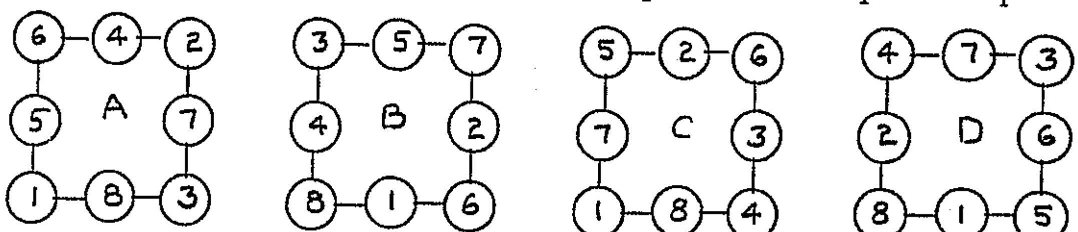


Figure 1

Subtract each entry in A from 9 to get the corresponding entry in B. Note that $3 \cdot 9 - 12 = 15$. Similarly examine C and D.

2. Four points on a plane determine six line segments. Figure 2 shows one way to place the four points so that the six line segments have just two different lengths. You get one point, up to a maximum of four, for each essentially different way (simply rotating or reflecting a figure doesn't count) you can place four points so that the six connecting line segments have just two different lengths. Clearly indicate the given four points, since the lines may cross in places not included among the given points, as in Figure 2.

3. To begin with, a student walks down a row of lockers, numbered from 1 to 500, making sure that they are all closed but not locked. Student 2 walks along the lockers and changes the position (in this case opens) all the doors with numbers divisible by 2. Student 3 then changes the position of the doors of lockers with numbers divisible by 3 (opening number 3, closing number 6, etc.). This process continues, with student n changing the position (opening closed doors, closing open doors) of doors on lockers with numbers divisible by n.

open ______ (a) What is the final position of door 360 (open or closed)?

12 ______ (b) How many times will door 360 have been opened when the process is complete?

69 4. A quadrilateral is formed by joining in order $O(0,0)$ to $A(16,0)$ to $B(8,14)$ to $C(2,10)$, and back to O. The midpoints of the sides of OABC are then joined in order to form a second quadrilateral. What is its area?

$\frac{(1-x)}{2(x^2-2x+2)}$ 5. In the square $ABCD$ with sides of length 1 shown in Figure 5, let $x = AE = BF = CG = DH$. Find the area of ΔBSE in terms of x.

6. Let x be the length of the base of an isosceles triangle having legs of length 10. We wish to study the function f that expresses the area A of the triangle as a function of x; i.e. $A = f(x)$.

$\frac{x}{4}\sqrt{400-x^2}$ (a) (2 points) Write a formula for $f(x)$

$0 \le x \le 20$ (b) (1 point) What is the domain of this function? Don't quibble about end points. Accept $0 < x < 20$

$10\sqrt{2}$ (c) (1 point) What is the value of x that maximizes A?

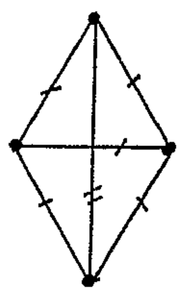


Figure 2

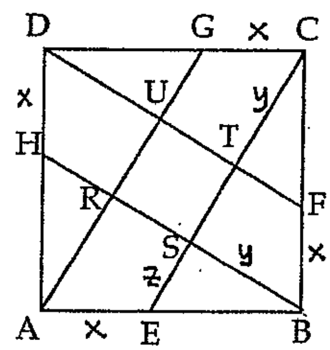
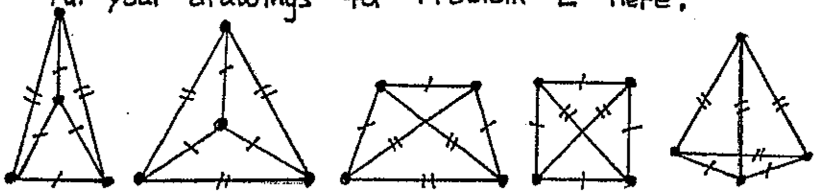


Figure 5

Put your drawings for Problem 2 here.



Team____________________

TEAM EVENT 5 SOLUTIONS

3. $360 = 2^3 3^2 \cdot 5$

The position of the door is changed for each number of the form

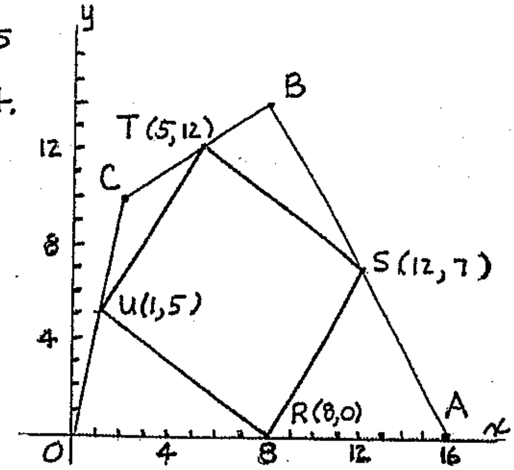
$2^{n_1}3^{n_2}5^{n_3}$, $0 \le n_1 \le 3$, $0 \le n_2 \le 2$, $0 \le n_3 \le 1$

excluding the case where $n_1 = n_2 = n_3 = 0$

$\therefore$ there are $4 \cdot 3 \cdot 2 - 1 = 23$ changes.

Of these, changes $1, 3, \ldots, 23 = \{2n-1 : n = 1, \ldots, 12\}$ open the door, so the door is opened 12 times, and the 23rd change leaves the door open.

4.



Midpoints are R, S, T, U as shown

slope UT = slope $RS = \frac{7}{4}$

and $UT = RS = \sqrt{7^2+4^2} = \sqrt{65}$

so RSTU is a parallelogram.

The line through U and T is

$y - 5 = \frac{7}{4}(x-1)$

or $-7x + 4y - 13 = 0$

Distance from R to TU is

$$\left|\frac{-7(8)+4(0)-13}{\sqrt{49+16}}\right| = \frac{69}{\sqrt{65}}$$

$$\text{Area}(RSTU) = \sqrt{65}\,\frac{69}{\sqrt{65}} = 69$$

5. Refer to Figure 5 on the Answer sheet for notation.

$\Delta BSE \sim \Delta CSB$, so $\dfrac{z}{1-x} = \dfrac{y}{1}$

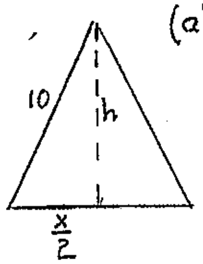
It was shown in Event 5B that

$$y = \frac{1-x}{\sqrt{x^2-2x+2}}, \text{ so}$$

$$z = (1-x)y = \frac{(1-x)^2}{\sqrt{x^2-2x+2}}$$

$$\text{Area}(\Delta BSE) = \frac{1}{2}yz = \frac{(1-x)^3}{2(x^2-2x+2)}$$

6.

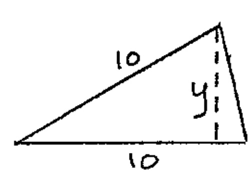


(a) $A = \frac{1}{2}xh$

$$h^2 = 100 - \frac{x^2}{4} = \frac{400-x^2}{4}$$

$$A = \frac{1}{2} \times \frac{\sqrt{400-x^2}}{2}$$

(b) See from the expression for A, or from the picture that x cannot exceed 20.



(c) Using one of the legs as a base and y as the corresponding height,

$A = \frac{1}{2}(10)y = \frac{y}{2}$. A is maximized when y is max.

But $y \le 10$. When $y = 10$, $x = 10\sqrt{2}$